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Collectively Compact Sets of Nonlinear Operators in Locally Convex Spaces

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Collectively compact sets of (linear) operators in Banach spaces have been studied and used by P. M. Anselone [2] and others in connection with integral operators. In this paper we show that a relevant part of the theory extends to bounded (in general nonlinear) operators in locally convex spaces.

1. INTRODUCTION

The concept of collective compactness of a set of (linear) operators in Banach spaces has been introduced by Anselone and Moore [4], where this notion has been proved useful in the study of integral equations. The subsequent investigations by Anselone and co-authors showed equally that the collectively compact and precompact sets of operators deserve interest of their own (see [2] for a detailed account and for further references, e.g., [3, 5, 14]). Here we are especially interested in the properties which relate these sets with the sets of corresponding adjoints [3, 14, 2, pp. 82–91]; these properties have been expressed for linear operators in Banach spaces. On the other hand, the concept of the adjoint for "$\mathscr{A}$-bounded" mappings with values in locally convex spaces has been defined and used for an extension of Schauder's and Gantmacher's theorem in [6]. Applications of the theory of $\mathscr{A}$-bounded compact mappings have been given in [6, 7, 9, 1]; further connected results, also in the direction of Banach–Steinhaus theorems for nonlinear mappings, have been proven in [10, 11, 12]. Here we show that the theory of collectively compact sets of (linear) operators in Banach spaces on one side and the theory of $\mathscr{A}$-bounded mappings in locally convex spaces on the other side are related and can both be used for the proof of new results which partly extend either theory towards the other (and in particular the classical results on linear compact mappings [15, pp. 153–154]). We close with an application to the Urysohn integral operator.

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2. Preliminaries

Let X denote a nonvoid set and $\mathscr{A}$ denote a system of nonempty subsets with union X such that for any A_1, $A_2 \in \mathscr{A}$ there exists $A \in \mathscr{A}$ with $A_1 \cup A_2 \subset A$. An operator f from X into a separated locally convex space Y is called $\mathscr{A}$-bounded [$\mathscr{A}$- (pre-) compact] if fA is bounded [contained in a (pre-) compact subset] in Y for every $A \in \mathscr{A}$; a set $\mathscr{K}$ of $\mathscr{A}$-bounded operators $f: X \to Y$ is called collectively $\mathscr{A}$- (pre-) compact if $\mathscr{K}A := \bigcup_{f\in\mathscr{K}} fA$ is contained in a (pre-) compact subset of Y (topological concepts are used in the sense of [15]). $\mathbb{K}$ being the scalar field, let E be a linear subspace of the $\mathbb{K}$-linear space $F_{\mathscr{A}}(X, \mathbb{K})$ of all $\mathscr{A}$-bounded mappings $e: X \to \mathbb{K}$. In E we consider the topology $T_{\mathscr{A}}$ of uniform convergence on the sets of $\mathscr{A}$; a base of neighborhoods is given by the sets ϵA^β, $\epsilon > 0$, $A \in \mathscr{A}$, where $A^\beta := \{e \in E: | ex | \leq 1 \text{ for } x \in A\}$ is the "β-polar" of $A \in \mathscr{A}$ [6, p. 6]. The pairing (X, E) is called a generalized dual system [10, p. 12] if E separates the points of X, that is, the mapping $\kappa^E: X \to E' := (E, T_{\mathscr{A}})'$, $x \mapsto \kappa^E x$, with $\langle \kappa^E x, e\rangle := ex$, $e \in E$, is injective. For an $\mathscr{A}$-bounded mapping $f: X \to Y$, the adjoint $f': Y' \to F_{\mathscr{A}}(X, \mathbb{K})$ is defined by $f'y' := y' \circ f$, $y' \in Y'$ [6, p. 9]. The adjoint is linear and $\beta(Y', Y) - T_{\mathscr{A}}$-continuous. We denote by $B_E(X, Y)$ the $\mathbb{K}$-linear space of $\mathscr{A}$-bounded mappings $f: X \to Y$ whose adjoints take their values in E. In this space the topology of uniform convergence on the sets of $\mathscr{A}$ is again denoted by $T_{\mathscr{A}}$. A base of neighborhoods is given by the absolutely convex and closed sets $W_{A,V} := \{f \in B_E(X, Y): fA \subset V\}$, $A \in \mathscr{A}$, $V \in \mathscr{V}$, where $\mathscr{V}$ is a base of absolutely convex and closed neighborhoods of zero in Y. If Y and E are complete then $(B_E(X, Y), T_{\mathscr{A}})$ is complete. For $f \in B_E(X, Y)$ we denote by f_A the restriction of f to $A \in \mathscr{A}$ and by f_A' the function $Y' \to E_A := \{e_A: e \in E\}$, $y' \to y' \circ f_A$. E_A is a normed space with $\| e_A \| := \sup\{| ex |: x \in A\}$. For a subset $\mathscr{K} \subset B_E(X, Y)$ and $A \in \mathscr{A}$ we define $\mathscr{K}_A := \{f_A: f \in \mathscr{K}\}$ and $\mathscr{K}_A' := \{f_A': f \in \mathscr{K}\}$. Let $L((Y', \beta), (E, T_{\mathscr{A}}))$ [$L((Y', \sigma), (E, \sigma))$] denote the $\mathbb{K}$-linear space of the continuous linear mappings $u: (Y', \beta(Y', Y)) \to (E, T_{\mathscr{A}})$ [$u: (Y', \sigma(Y', Y'')) \to (E, \sigma(E, E'))$] and $T_{\mathscr{C}}$ the topology of uniform convergence on the system $\mathscr{C}$ of equicontinuous subsets of Y'. A neighborhood base of absolutely convex and closed sets in $T_{\mathscr{C}}$ is given by the sets $N_{V^\circ, A^\beta} := \{u: uV^\circ \subset A^\beta\}$, $A \in \mathscr{A}$, $V \in \mathscr{V}$. We remark that either mapping ψ,

$$\psi_\beta: (B_E(X, Y), T_{\mathscr{A}}) \to (L((Y', \beta), (E, T_{\mathscr{A}})), T_{\mathscr{C}}), \qquad f \mapsto f',$$
$$\psi_\sigma: (B_E(X, Y), T_{\mathscr{A}}) \to (L((Y', \sigma), (E, \sigma)), T_{\mathscr{C}}), \qquad f \mapsto f',$$

is a topological isomorphism on the range R of ψ. In fact, ψ is well defined and linear, $\psi f = 0$ implies $f = 0$, and ψ is continuous and relatively open because for $A \in \mathscr{A}$, $V \in \mathscr{V}$ we have

$$\begin{aligned}\psi(W_{A,V}) &= \{f': f \in B_E(X, Y), fA \subset V\}\\ &= \{f': f \in B_E(X, Y), f'V^\circ \subset A^\beta\}\\ &= N_{V^\circ, A^\beta} \cap R.\end{aligned}$$

For further details see [6, 10, 11].

3. Precompact Sets of $\mathscr{A}$-Precompact Operators in $(B_E(X, Y), T_{\mathscr{A}})$

Palmer [14] proved that for normed spaces X, Y, a subset $\mathscr{K} \subset L(X, Y)$ is a precompact set of precompact operators iff $\mathscr{K}$ and $\mathscr{K}'$ are collectively precompact, iff $\mathscr{K}X_1$ is precompact and $\mathscr{K}'y'$ is precompact for each $y' \in Y_1'$, or iff $\mathscr{K}x$ is precompact for each $x \in X_1$ and $\mathscr{K}'Y_1'$ is precompact (the subscript 1 denoting the unit ball in the corresponding space). We use the nonlinear version of Schauder's theorem [6, p. 13] to get the following extension of his results.

Theorem 1. *For a subset $\mathscr{K}$ of $B_E(X, Y)$ the following conditions are equivalent:*

(a) *$\mathscr{K}$ is a $T_{\mathscr{A}}$-precompact set of $\mathscr{A}$-precompact operators.*

(b) *$\mathscr{K}'$ is a $T_{\mathscr{C}}$-precompact set of $\mathscr{C}$-compact operators in $L(Y', E)$.*

(c) *$\mathscr{K}$ is collectively $\mathscr{A}$-precompact and $\mathscr{K}'$ is collectively $\mathscr{C}$-precompact.*

(d) *$\mathscr{K}$ is collectively $\mathscr{A}$-precompact and for all $y' \in Y'$ the set $\mathscr{K}'y'$ is precompact in E.*

(e) *$\mathscr{K}'$ is collectively $\mathscr{C}$-precompact and for all $x \in X$ the set $\mathscr{K}x$ is precompact in Y.*

Proof. It follows from the properties of ψ that $\mathscr{K}$ is $T_{\mathscr{A}}$-precompact iff $\mathscr{K}'$ is $T_{\mathscr{C}}$-precompact. Furthermore, by [6, p. 13] an element in $B_E(X, Y)$ is $\mathscr{A}$-precompact iff its adjoint is $\mathscr{C}$-compact. Hence we have (a) $\Leftrightarrow$ (b). Now assume (a) and let $A \in \mathscr{A}$, $V \in \mathscr{V}$ be given. There are elements $f_1, \ldots, f_n \in \mathscr{K}$ such that $\mathscr{K} \subset \bigcup_{i=1}^{n} (f_i + W_{A,V})$. Hence $\mathscr{K}A \subset \bigcup_{i=1}^{n} (f_i A + V)$. Because $f_1, \ldots, f_n$ are $\mathscr{A}$-precompact there are elements $y_1, \ldots, y_m \in \bigcup_{i=1}^{n} f_i A$ with $\bigcup_{i=1}^{n} f_i A \subset \bigcup_{j=1}^{m} (y_j + V)$. It follows that $\mathscr{K}A \subset \bigcup_{j=1}^{m} (y_j + 2V)$, so that (a) implies the first part of (c). Similarly (b) implies the second part, so that (a) $\Leftrightarrow$ (b) $\Rightarrow$ (c). Because $\mathscr{A}$ covers X and $\mathscr{C}$ covers Y' we have (c) $\Rightarrow$ (d) and (c) $\Rightarrow$ (e). Now we prove (e) $\Rightarrow$ (a). Let $A \in \mathscr{A}$, $V \in \mathscr{V}$ be given. The completion $\hat{E}$ of $(E, T_{\mathscr{A}})$ is the closure $\bar{E}$ of E in the complete space $(F_{\mathscr{A}}(X, \mathbb{K}), T_{\mathscr{A}})$ [10, p. 29]. We can identify $(\hat{E}, T_{\mathscr{A}})' = (\bar{E}, T_{\mathscr{A}})'$ with E' [15, p. 108]; each element of $(\hat{E}, T_{\mathscr{A}})'$ being a continuous extension of an element in E'. By assumption, $\mathscr{K}'V^{\circ}$ is precompact in $(E, T_{\mathscr{A}})$, hence also in $(\bar{E}, T_{\mathscr{A}})$ and the closure S of $\mathscr{K}'V^{\circ}$ in $(\bar{E}, T_{\mathscr{A}})$ is compact. Because $\kappa^{\bar{E}}A \subset A^{\beta\circ}$ (A^{β} taken in $\bar{E}$, $A^{\beta\circ}$ taken in E') $\kappa^{E}A$ is equicontinuous on $\bar{E}$. Hence $\kappa^{E}A\,|_S$ is a bounded set of equicontinuous functions on $(S, T_{\mathscr{A}} \cap S)$. By Arzelà–Ascoli's theorem, there exist elements $a_1, \ldots, a_n \in A$ such that for all $e' \in \kappa^{E}A$,

$$\inf_{i \in \{1,\ldots,n\}} \sup_{s \in S} |\langle e', s\rangle - \langle \kappa^{E}a_i, s\rangle| \leqslant 1.$$

In particular we have for all $x \in A$,

$$\inf_{i \in \{1,\ldots,n\}} \sup_{\substack{y' \in V^{\circ} \\ f \in \mathscr{K}}} |\langle \kappa^{E}x - \kappa^{E}a_i, f'y'\rangle| \leqslant 1.$$

For each $i \in \{1,..., n\}$ there exists by assumption a finite set $\{f_{i,1},..., f_{i,k_i}\} \subset \mathscr{K}$ such that

$$\mathscr{K} a_i \subset \bigcup_{j=1}^{k_i} (f_{i,j} a_i + V).$$

This implies that for each $i \in \{1,..., n\}$ we have for all $f \in \mathscr{K}$

$$\inf_{j \in \{1,\ldots,k_i\}} \sup_{y' \in V^\circ} |\langle y', f a_i - f_{i,j} a_i \rangle| \leqslant 1.$$

The following estimate holds for all $x \in A$, $f \in \mathscr{K}$, $y' \in V^\circ$:

$$\begin{aligned} &|\langle fx - f_{i,j}x, y' \rangle| \\ &\quad \leqslant |\langle \kappa^E x - \kappa^E a_i, f' y' \rangle| + |\langle \kappa^E x - \kappa^E a_i, f'_{i,j} y' \rangle| + |\langle y', f a_i - f_{i,j} a_i \rangle|; \end{aligned}$$

hence we have for every $f \in \mathscr{K}$

$$\inf_{\substack{i \in \{1,\ldots,n\} \\ j \in \{1,\ldots,k_i\}}} \sup_{\substack{x \in A_c \\ y' \in V^\circ}} |\langle fx - f_{i,j}x, y' \rangle| \leqslant 3,$$

which means that $\mathscr{K} \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{k_i} (f_{i,j} + W_{A,3V})$, so that $\mathscr{K}$ is $T_{\mathscr{A}}$-precompact. Because $\mathscr{K}'$ is collectively $\mathscr{C}$-precompact, each element $f' \in \mathscr{K}'$ is $\mathscr{C}$-precompact. It follows that each $f \in \mathscr{K}$ is $\mathscr{A}$-precompact (see the proof of (b) $\Rightarrow$ (a) in [6, Theorem 5]). This proves (e) $\Rightarrow$ (a), and (d) $\Rightarrow$ (b) is proved in a similar way.

4. $\mathscr{A}$-Weakly Compact Operators in $B_E(X, Y)$

If Y is endowed with its weak topology $\sigma(Y, Y')$ the $\mathscr{A}$-precompact mappings coincide with the $\mathscr{A}$-bounded mappings. To investigate the validity of a corresponding result for $\mathscr{A}$-weakly compact operators we first improve upon an earlier extension of Gantmacher's Theorem [6, Theorem 4, p. 12]. Let us recall that an $\mathscr{A}$-bounded mapping $f: X \to Y$ is called $\mathscr{A}$-weakly compact if fA is contained in an absolutely convex $\sigma(Y, Y')$-compact subset of Y for $A \in \mathscr{A}$.

Theorem 2. *Let $f \in B_E(X, Y)$ and consider the following statements:*

(α) *f is $\mathscr{A}$-weakly compact.*

(β) *f' maps the $\beta(Y', Y)$-bounded sets into $\sigma(E, E')$-compact sets.*

Then the following holds: If

(1) *every absolutely convex $\beta(Y', Y)$-bounded set in Y' is relatively $\sigma(Y', Y)$-compact, we have* (α) $\Rightarrow$ (β). *If*

(2) *Y is closed in $(Y'', \beta(Y'', Y'))$, we have* (β) $\Rightarrow$ (α).

Proof. If (α) holds, f' is $\sigma(Y', Y) - \sigma(E, E')$-continuous [6, Theorem 1, p. 11]. Hence (β) follows from (1). Now assume (β). Because f' is $\sigma(Y', Y'') - \sigma(E, E')$-continuous and maps the $\sigma(Y', Y'')$-bounded (equivalently the $\beta(Y', Y)$-bounded) sets into absolutely convex $\sigma(E, E')$-compact sets, $f'' \colon (E', \tau(E', E)) \to (Y'', \beta(Y'', Y'))$ is continuous [15, p. 48] and therefore $\sigma(E', E) - \sigma(Y'', Y''')$-continuous. The $\sigma(Y'', Y''')$-closure $\overline{\mathrm{co}\, fA}$ of the absolutely convex hull of fA, $A \in \mathscr{A}$, coincides with the $\beta(Y'', Y')$-closure and is by (2) contained in Y. On the other hand, $\overline{\mathrm{co}\, fA}$ is contained in $f''A^{\beta^\circ}$, which is $\sigma(Y'', Y''')$-compact by the continuity of f'' and the $\sigma(E', E)$-compactness of A^{β°. Hence $\overline{\mathrm{co}\, fA}$ is $\sigma(Y'', Y''')$-(equivalently $\sigma(Y'', Y''') \cap Y$-) compact. Because $\sigma(Y'', Y''') \cap Y$ is finer than $\sigma(Y, Y')$, $\overline{\mathrm{co}\, fA}$ is $\sigma(Y, Y')$-compact. This proves (α).

V. Krishnamurthy has termed the separated locally convex spaces with property (1) "quasi-M-barrelled" [13, p. 337]. He has proved that a separated locally convex space Y possesses the properties (1) and (2) iff Y is minimal in Y'', which means by definition that Y is $\beta(Y'', Y')$-closed in Y'', that Y is $\sigma(Y'', Y')$-dense in Y'' and that no proper subspace of Y has these last properties [13, p. 338].

COROLLARY. *If Y is minimal in Y'' and complete, a member $f \in B_E(X, Y)$ is $\mathscr{A}$-weakly compact iff f' is $\mathscr{C}$-weakly compact.*

Proof. In view of Theorem 2 we only need to show that under the given assumptions (β) holds iff f' maps the equicontinuous sets into absolutely convex $\sigma(E, E')$-compact subsets. Because each equicontinuous subset of Y' is $\beta(Y', Y)$-bounded the necessity of the last condition is obvious. To prove its sufficiency, we remark that the condition and the completeness of Y imply that f' is $\sigma(Y', Y) - \sigma(E, E')$-continuous [6, Theorem 3, p. 12], and (β) follows from (1).

THEOREM 3. *Let Y be minimal in Y'' and let Y and E be complete. For a subset $\mathscr{K}$ of $B_E(X, Y)$ the following conditions are equivalent:*

(a′) *$\mathscr{K}$ is a $T_{\mathscr{A}}$-precompact set of $\mathscr{A}$-weakly compact operators.*

(b′) *$\mathscr{K}'$ is a $T_{\mathscr{C}}$-precompact set of $\mathscr{C}$-weakly compact operators in $L(Y', E)$.*

In this case we have

(c′) *$\mathscr{K}$ is collectively $\mathscr{A}$-weakly compact (i.e., $\bigcup_{f \in \mathscr{K}} \mathrm{co}\, fA$ is relatively weakly compact for $A \in \mathscr{A}$), and $\mathscr{K}'$ is collectively $\mathscr{C}$-weakly compact.*

Proof. The properties of ψ and the corollary of Theorem 2 imply the equivalence (a′) $\Leftrightarrow$ (b′). To show that (a′) implies the first part of (c′) let $A \in \mathscr{A}$ be given and consider a net $y_i := \sum_{k=1}^{k_i} \lambda_k^i f_i(x_k^i)$ ($f_i \in \mathscr{K}$, $\sum_{k=1}^{k_i} |\lambda_k^i| \leqslant 1$, $x_k^i \in A$) in $\bigcup_{f \in \mathscr{K}} \mathrm{co}\, fA$. $\mathscr{K}$ being precompact in the complete space $B_E(X, Y)$, the net $\{f_i\}$ contains a $T_{\mathscr{A}}$-convergent subnet $f_{i_\alpha} \to f$. The limit f is $\mathscr{A}$-weakly compact [6, Theorem 8, p. 16] and the net $z_{i_\alpha} := \sum_{k=1}^{k_{i_\alpha}} \lambda_k^{i_\alpha} f(x_k^{i_\alpha})$ again contains a $\sigma(Y, Y')$-

convergent subnet $z_{i_{\alpha\beta}} \to z \in Y$. It then follows that $y_{i_{\alpha\beta}} \to z$, and $\mathscr{K}$ is collectively $\mathscr{A}$-weakly compact. Similarly, (b') implies the second part of (c').

Remark. Uniformly bounded sets of linear bounded operators in reflexive Banach spaces of infinite dimension show that (c') does not imply (a'), (b') in general. Furthermore, either condition (d'), (e') (obtained from (d), (e) by replacing "$\mathscr{A}$- ($\mathscr{C}$-) precompact" by "$\mathscr{A}$- ($\mathscr{C}$-) weakly compact" and "precompact" by "relatively weakly compact") is implied by (c') but does not imply (c') in general; also (d'), (e') are in general incomparable.

5. Collectively $\mathscr{A}$-Precompact Subsets of $B_E(X, Y)$

In [14] Palmer gave necessary and sufficient conditions for a subset $\mathscr{K} \subset L(X, Y)$ (X, Y normed spaces) or for the set $\mathscr{K}'$ of adjoints to be collectively (pre-) compact. One part of his results can be expressed for $\mathscr{A}$-bounded operators as follows.

THEOREM 4. *Let Y be a normed space and $\mathscr{K}$ be a subset of $B_E(X, Y)$. Then $\mathscr{K}$ is collectively $\mathscr{A}$-precompact iff $\mathscr{K}$ is bounded in $(B_E(X, Y), T_{\mathscr{A}})$ and for each $\epsilon > 0$ and $A \in \mathscr{A}$ there is a closed subspace $Z(\epsilon, A)$ of finite codimension in Y' such that $\| f'_A |_{Z(\epsilon, A)} \| \leqslant \epsilon$ for all $f \in \mathscr{K}$.*

The proof makes use of an element of the linear theory [14, Theorem 3.2, p. 103].

6. An Example

We review the Urysohn integral operator $f\colon C(S) \to C(Q)$ (considered in [7, p. 175]) which is given by

$$fx(q) = \int_S K(q, t, x(t))\, d\lambda(t), \qquad x \in C(S), \quad q \in Q.$$

Here S, Q are compact Hausdorff spaces and $C(S)$, $C(Q)$ are the corresponding Banach spaces of $\mathbb{R}$-valued continuous functions. Let M_α be the Banach space of the $\mathbb{R}$-valued continuous functions φ on $[-\alpha, +\alpha]$ with $\varphi(0) = 0$, $\alpha > 0$, and M the locally convex space of the $\mathbb{R}$-valued continuous functions on $\mathbb{R}$ with $\varphi(0) = 0$, endowed with the topology of compact convergence; for $\varphi \in M_\alpha$, let $D_\delta\varphi$ denote the modulus of continuity of φ and $\varphi_\alpha := \varphi |_{[-\alpha, +\alpha]}$ for $\varphi \in M$. Then the assumptions on K and λ are as follows:

(i) λ is a positive regular measure on the σ-algebra $\mathscr{B}$ of Borel sets in S with $\lambda(S) < \infty$,

(ii) for every $q \in Q$ and λ-almost all $t \in S$, $K(q, t)\colon z \mapsto K(q, t, z)$ is an element of M and $K(q, \cdot)_\alpha \in L^1_{M_\alpha}(S, \mathscr{B}, \lambda)$ for $\alpha > 0$,

(iii) $\sup_{q \in Q} \int_S \| K(q, t)_\alpha \| \, d\lambda(t) < \infty$ and $\lim_{\delta \to 0} \sup_{q \in Q} \int_S D_\delta K(q, t)_\alpha \, d\lambda(t) = 0$, $\alpha > 0$,

(iv) $\int_B K(q, t, z) \, d\lambda(t)$ is continuous in $q \in Q$ for all $B \in \mathscr{B}$, $z \in \mathbb{R}$.

Let $\{p_j\}_{j=1}^\infty$ be a basis of polygonal approximation in M and $\{\nu_j\}_{j=1}^\infty$ a corresponding sequence of coefficient functionals in M' (which are linear combinations of evaluation functionals). If

$$s_n \varphi := \sum_{j=1}^{n} \langle \nu_j , \varphi \rangle \, p_j , \qquad \varphi \in M, \quad n \in \mathbb{N},$$

then $(s_n \varphi)_\alpha \to \varphi_\alpha$ in M_α and $\kappa_\alpha := \sup_n \| s_n |_{M_\alpha} \| < \infty$, $\alpha > 0$. For $n \in \mathbb{N}$ let us define $f_n\colon C(S) \to C(Q)$ by

$$f_n x(q) := \int_S (s_n K(q, t)) \, (x(t)) \, d\lambda(t), \qquad x \in C(S), \quad q \in Q.$$

Let $X := C(S)$, $\mathscr{A} := \{A_\alpha\}_{\alpha > 0}$, $A_\alpha := \{x \in X \colon \| x \| \leqslant \alpha\}$, $Y := C(Q)$, and let E be the space of functionals $e\colon X \to \mathbb{R}$ to which there corresponds a function $g\colon S \to M$ (λ-almost everywhere) with $g(\cdot)_\alpha \in L^1_{M_\alpha}(S, \mathscr{B}, \lambda)$, $\alpha > 0$, such that $ex = \int_S g(t, x(t)) \, d\lambda(t)$, $x \in C(S)$; that is, to E_{A_α} there corresponds a space of M_α-valued Bochner-integrable functions. $(E, T_{\mathscr{A}})$ is complete, and for each $\mu \in Y' = rca(Q)$ [8, p. 265] $f_n' \mu$ corresponds to a function g_μ of the separated form $g = \sum_{j=1}^{n} h_{j,\mu} p_j$, $h_{j,\mu} \in L^1(S, \mathscr{B}, \lambda)$, so that $f_n \in B_E(X, Y)$, $n \in \mathbb{N}$. Let $\alpha > 0$. We prove $f_n x \to fx$ uniformly for $x \in A_\alpha$. Given $\epsilon > 0$, (iii) implies that there exists $\eta > 0$ such that for every $B \in \mathscr{B}$ with $\lambda(B) < \eta$ we have

$$\sup_{q \in Q} \int_B \| K(q, t)_\alpha \| \, d\lambda(t) \leqslant \epsilon / \kappa_\alpha$$

and there exists a compact set $C_\eta \subset M_\alpha$ and for every $q \in Q$ a set $S_{q,\eta} \in \mathscr{B}$ with $\lambda(S_{q,\eta}) < \eta$ such that

$$\{K(q, t)_\alpha \colon q \in Q, \, t \notin S_{q,\eta}\} \subset C_\eta$$

(see [7, pp. 170–171]). Then $s_n \varphi \to \varphi$ uniformly on C_η and there exists $n(\epsilon)$ such that for $n \geqslant n(\epsilon)$ we have $\| s_n \varphi - \varphi \| \leqslant \epsilon / \lambda(S)$ for $\varphi \in C_\eta$. It follows for $x \in A_\alpha$, $n \geqslant n(\epsilon)$, $q \in Q$,

$$| f_n x(q) - fx(q) | \leqslant \int_{S_{q,\eta}} \| (s_n K(q, t))_\alpha - K(q, t)_\alpha \| \, d\lambda(t)$$

$$+ \int_{S \setminus S_{q,\eta}} \| s_n K(q, t)_\alpha \| \, d\lambda(t) + \int_{S \setminus S_{q,\eta}} \| K(q, t)_\alpha \| \, d\lambda(t) \leqslant 3\epsilon.$$

Hence $f_n \to f$ in $B_E(X, Y)$ and $\mathscr{K} := \{f_n\}_{n\in\mathbb{N}}$ is a $T_{\mathscr{A}}$-precompact subset of $B_E(X, Y)$. To see that the situations (a′), (b′) hold, one can either write $f_n = \sum_{j=1}^n L_j \circ P_j$ and observe that the mappings $P_j\colon C(S) \to C(S)$, $x \mapsto p_j \circ x$ are bounded and the linear mappings $L_j\colon C(S) \to C(Q)$, $x \mapsto (q \mapsto \int_S \langle \nu_j, K(q, t)\rangle\, x(t)\, d\lambda(t))$ are weakly compact [8, p. 493] or else one can show the weak compactness of $f_n'\colon C(Q)' \to E$ by noting that it maps the unit ball B in $rca(Q)$ into a set of functions $\{\sum_{j=1}^n h_{j,\mu} p_j\colon \mu \in B\}$, where $\{h_{j,\mu}\colon \mu \in B\}$ is a bounded set of uniformly integrable functions in $L^1(S, \mathscr{B}, \lambda)$, and use the classical Pettis criterion for weak compactness in $L^1_{M_\alpha{}^n}(S, \mathscr{B}, \lambda)$ ($M_\alpha{}^n := \overline{\operatorname{span}}\{p_{1\alpha}, \ldots, p_{n\alpha}\}$ finite dimensional). In either case, one obtains (a′), (b′) by using parts of Theorem 3. It follows from (a′) that f is weakly compact. We remark that f is the most general (nonzero) weakly compact operator from $C(S)$ into $C(Q)$ which is uniformly continuous on bounded sets and satisfies the algebraic relation $T(x + x_1 + x_2) = T(x + x_1) + T(x + x_2) - Tx$, for $x, x_1, x_2 \in C(S)$ with x_1, x_2 having disjoint support, and $T0 = 0$ [7].

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